

HADWIGER'S CONJECTURE: FINITE VS INFINITE GRAPHS

DOMINIC VAN DER ZYPEN

ABSTRACT. We study some versions of the statement of Hadwiger's conjecture for finite as well as infinite graphs.

1. DEFINITIONS

In this note we are only concerned with simple undirected graphs $G = (V, E)$ where V is a set and $E \subseteq \mathcal{P}_2(V)$ where

$$\mathcal{P}_2(V) = \{\{x, y\} : x, y \in V \text{ and } x \neq y\}.$$

We denote the vertex set of a graph G by $V(G)$ and the edge set by $E(G)$. Moreover, for any cardinal α we denote the complete graph on α points by K_α .

For any graph G , disjoint subsets $S, T \subseteq V(G)$ are said to be *connected to each other* if there are $s \in S, t \in T$ with $\{s, t\} \in E(G)$.

Given a collection \mathcal{D} of pairwise disjoint, nonempty, connected subsets of V , we associated with \mathcal{D} a graph $G(\mathcal{D})$ with vertex set \mathcal{D} and

$$E(G(\mathcal{D})) = \{\{d, e\} : d \neq e \in \mathcal{D} \text{ and } d, e \text{ are connected to each other}\}.$$

We say that a graph M is a *minor* of a graph G if there is a collection \mathcal{D} of pairwise disjoint, nonempty, connected subsets of V and an injective graph homomorphism $f : M \rightarrow G(\mathcal{D})$.

This implies that K_α is a *minor* of a graph G if and only if there is a collection $\{S_\beta : \beta \in \alpha\}$ of nonempty, connected and pairwise disjoint subsets of $V(G)$ such that for all $\beta, \gamma \in \alpha$ with $\beta \neq \gamma$ the sets S_β and S_γ are connected to each other.

2. DIFFERENT STATEMENTS OF HADWIGER'S CONJECTURE

The statement of Hadwiger's conjecture that is usually found in the literature is the following:

(H): If G is a simple undirected graph and $\lambda = \chi(G)$ then the complete graph K_λ is a minor of G .

The next version of Hadwiger's statement has a bit of a different flavor, and we will compare it to (H) in the finite and infinite contexts in the following sections.

(ModH): For every graph G there is a minor M of G such that

- (1) $M \not\cong G$, and
- (2) $\chi(M) = \chi(G)$.

There is a version of (ModH) that has appears to be similar, but we will see later that it is worthwhile to look at the statement separately.

(HomH): For every graph G there is a minor M of G such that

- (1) $M \not\cong G$, and
- (2) there is a graph homomorphism $f : G \rightarrow M$.

Last, the following weaker version of this was studied in [4]:

(WeakH): Whenever λ is a cardinal such that there is no graph homomorphism $c : G \rightarrow K_\lambda$ then K_λ is a minor of G .

3. THE FINITE CASE

Overview:

- (H) is a long-standing open problem.
- (ModH) is equivalent to (H) for finite graphs (see proposition 3.1).
- (HomH) is also equivalent to (H) for finite graphs.
- (WeakH) is implied by (H).

Proposition 3.1. *For finite graphs G , the statements (H) and (ModH) are equivalent.*

Proof. Given a finite non-complete graph $G = (V, E)$, the statement (H) implies that $K = K_{\chi(G)}$ is a minor of G . Since K is complete, but not G , they are not isomorphic, so (ModH) holds.

For the other implication, take any finite graph G and let $n = \chi(G)$. Use (ModH) to get a proper minor M_1 such that $\chi(M_1) = n$. If M_1 is complete, we have proved (H), otherwise use (ModH) again to find a proper minor M_2 of M_1 with $\chi(M_2) = n$, and so on. Since G is finite, this procedure is bound to end at some M_k for some $k \in \mathbb{N}$, which implies that M_k is complete and has n points. \square

It is easy to modify Proposition 3.1 to see that in the finite case, (H) and (HomH) are equivalent.

In the finite setting, the statement (WeakH) amounts to saying that if $\chi(G) = t > 0$ then K_{t-1} is a minor of G . This is weaker than (H); whether it is strictly weaker is an open question (see section 5).

4. THE INFINITE CASE

4.1. Infinite chromatic number. Overview:

- (H) is **false**: Let G be the disjoint union of all $K_n, n \in \mathbb{N}$. Then $\chi(G) = \omega$, but K_ω is not a minor of G .
- (ModH) is **true**, see proposition 4.1.
- (HomH) is open.
- (WeakH) is **true**, see [4].

So that is why we separately introduced (HomH) in addition to (ModH): they might be different for graphs with infinite chromatic number.

Proposition 4.1. *For graphs with infinite chromatic number, (ModH) is true.*

Proof. Let I be the set of isolated vertices of G .

Case 1. $I \neq \emptyset$. We set $M = G \setminus I$. It is easy to see that $M \not\cong G$ as M contains no isolated points. Since $\chi(G) \geq \aleph_0$ we have $\chi(M) = \chi(G)$.

Case 2. $I = \emptyset$. Fix $v_0 \in V(G)$. Let $M = (V(G), E)$ where

$$E = \{e \in E(G) : v_0 \notin e\},$$

that is we remove all edges connecting v_0 to some other vertex in $V(G)$. Since M has v_0 as an isolated point, but G has no isolated points, we have $M \not\cong G$, and it is easy to verify that $\chi(M) = \chi(G)$. \square

4.2. Finite chromatic number. For infinite graphs with finite chromatic number we get the following results:

- It is not known whether (H) and (WeakH) are true;
- (ModH) is **true**: the theorem of De Bruijn and Erdős [1] implies that if G is infinite with finite chromatic number, there is a finite subgraph M of G with $\chi(M) = \chi(G)$.
- (HomH) is true for the same reason (note that a coloring is always a graph homomorphism to a complete graph).

5. OPEN QUESTIONS

Question 1. Does the weak Hadwiger conjecture (WeakH) hold for finite graphs?

(WeakH) might be as elusive as (H) has been so far; so here is a different problem:

Question 2. When we restrict ourselves to finite graphs, does the weak Hadwiger conjecture (WeakH) imply the statement of the Hadwiger conjecture?

The next question is a stronger version of (ModH) and focuses on finite graphs.

Question 3. Suppose that G is a finite, connected graph such that whenever you contract 1 edge or 2 edges, the chromatic number decreases. Does this imply G is complete?

Finally we turn to infinite graphs:

Question 4. Does (WeakH) hold for infinite graphs with finite chromatic number?

Question 5. Does (HomH) hold for graphs with infinite chromatic number?

6. ACKNOWLEDGEMENT

I would like to thank user `@bof` from `mathoverflow.net` for his argument used in proposition 4.1 [5].

REFERENCES

- [1] de Bruijn Nicolaas, Erdős Paul, *A colour problem for infinite graphs and a problem in the theory of relations*, Nederl. Akad. Wetensch. Proc. Ser. A, **53** (1951), 371–373.
- [2] Hadwiger Hugo, *Über eine Klassifikation der Streckenkomplexe*, Vierteljschr. Naturforsch. Ges. Zürich, **88** (1943), 133–143.
- [3] Robertson Neil, Seymour Paul, Thomas Robin, *Excluding subdivisions of infinite cliques*, Trans. Amer. Math. Soc. (**332**) (1992), no. 1, 211–223.
- [4] van der Zypen Dominic, *A weak form of Hadwiger’s conjecture*, SOP Trans. on Applied Mathematics **1** (2014), no. 2, 84–87.
- [5] <http://mathoverflow.net/a/221663/8628>

FEDERAL OFFICE OF SOCIAL INSURANCE, EFFINGERSTRASSE 20, CH-3003 BERN, SWITZERLAND

E-mail address: dominic.zypen@gmail.com